The Problem of Points: The first expectation

Unit 1 Lecture 1

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Learning Objectives

After this lecture, you will be able to:

1. Describe the Problem of Points and the solution proposed by Pascal and Fermat.
2. Construct a tree diagram and use it to calculate the expected value.
3. Explain in what sense the expected value is a fair representation of the outcome of a game or experiment.
4. Graph a simple tree diagram using `ggtree`. See Appendix for `R` code.

These slides use the following `R` packages

Setup:

```r
library("tidyverse")
library("treeio")
library("ggtree")
theme_set(theme_bw())
```

The package `ggtree` is not available on the Comprehensive R Archive Network (CRAN). Install it from Bioconductor:

```r
install.packages("BiocManager")
BiocManager::install("ggtree")
```

The Problem of Points

- Pascal and Fermat (1654) were asked to solve a version of the following problem:
  - Two gamblers bet over a "fair" coin.
  - Player 1 is awarded one point if the coin lands on heads (H). Player 2 if the coin lands tails (T).
  - Best 2 out of 3 wins \$100. But the game is interrupted after the first flip. What is a fair way to divide the pot?

- Pascal and Fermat worked on the problem in a series of letters, which was how science was done before academic journals.
  - They calculated the expected winnings—the average amount won if the game were played to completion over and over again.

- Their correspondence—less than three thousand words in total—not only solved the then 160-year-old problem. It initiated the fields of probability and statistics.
  - Today, scientists largely rely on their reasoning to create and evaluate algorithms.

Blaise Pascal (1623-1662) Pierre de Fermat (1607-1665)

Source: https://en.wikipedia.org/wiki/Blaise_Pascal#/media/File:Blaise_Pascal_Versailles.jpg
https://en.wikipedia.org/wiki/Pierre_de_Fermat#/media/File:Pierre_de_Fermat.jpg

Part of Pascal and Fermat's correspondence (Tannery 1894)

IV. FIELD OF PROBABILITY

Fermat and Pascal on Probability

(Translated from the French by Professor Vera Sanford, Western Reserve University, Cleveland, Ohio.)

Italian writers of the fifteenth and sixteenth centuries, notably Pacioli (1494), Tartaglia (1556), and Cardan (1545), had discussed the problem of the division of a stake between two players whose game was interrupted before its close. The problem was proposed to Pascal and Fermat, probably in 1654, by the Chevalier de Méré, a gambler who is said to have had unusual ability "even for the mathematics." The correspondence which ensued between Fermat and Pascal, was fundamental in the development of modern concepts of probability, and it is unfortunate that the introductory letter from Pascal to Fermat is no longer extant. The one here translated, written in 1654, appears in the *Œuvres de Fermat* (ed. Tannery and Henry, Vol. II, pp. 288–314, Paris, 1894) and serves to show the nature of the problem. For a biographical sketch of Fermat, see page 213; of Pascal, page 67. See also pages 165, 213, 214, and 326.

Monsieur,

If I undertake to make a point with a single die in eight throws, and if we agree after the money is put at stake, that I shall not cast the first throw, it is necessary by my theory that I take $\frac{1}{6}$ of the total sum to be impartial because of the aforesaid first throw.

And if we agree after that that I shall not play the second throw, I should, for my share, take the sixth of the remainder that is $\frac{5}{36}$ of the total.

If, after that, we agree that I shall not play the third throw, I should to recoup myself, take $\frac{1}{6}$ of the remainder which is $\frac{25}{216}$ of the total.

And if subsequently, we agree again that I shall not cast the fourth throw, I should take $\frac{1}{6}$ of the remainder or $\frac{125}{1296}$ of the total, and I agree with you that that is the value of the fourth throw supposing that one has already made the preceding plays.

But you proposed in the last example in your letter (I quote your very terms) that if I undertake to find the six in eight throws and if I have thrown three times without getting it, and if my opponent

the given points, and leaves on the planes segments in which given angles may be inscribed;"[1] and this one: "Any three circles, any three points, and any three lines being given, to find a circle which touches the circles and the points and leaves on the lines an arc in which a given angle may be inscribed."

I solved these problems in a plane, using nothing in the construction but circles and straight lines, but in the proof I made use of solid loci,[2]—of parabolas, or hyperbolas. Nevertheless, inasmuch as the construction is in a plane, I maintain that my solution is plane, and that it should pass as such.

This is a poor recognition of the honor which you have done me in putting up with my discourse which has been plaguing you so long. I never thought I should say two words to you and if I were to tell you what I have uppermost in my heart,—which is that the better I know you the more I honor and admire you,—and if you were to see to what degree that is, you would allot a place in your friendship for him who is, Monsieur, your etc.

Pascal to Fermat
Monday, August 24, 1654

Monsieur,

1. I was not able to tell you my entire thoughts regarding the problem of the points by the last post,[3] and at the same time, I have a certain reluctance at doing it for fear lest this admirable harmony which obtains between us and which is so dear to me should begin to flag, for I am afraid that we may have different opinions on this subject. I wish to lay my whole reasoning before you, and to have you do me the favor to set me straight if I am in error or to indorse me if I am correct. I ask you this in all faith and sincerity for I am not certain even that you will be on my side.

When there are but *two* players, your theory which proceeds by combinations is very just. But when there are three, I believe I have a proof that it is unjust that you should proceed in any other manner than the one I have. But the method which I have disclosed to you and which I have used universally is common to all imaginable conditions of all distributions of points, in the place of that of combinations (which I do not use except in partic-

[1] ["...capable d'angles donnés."]
[2] [A common name for conics.]
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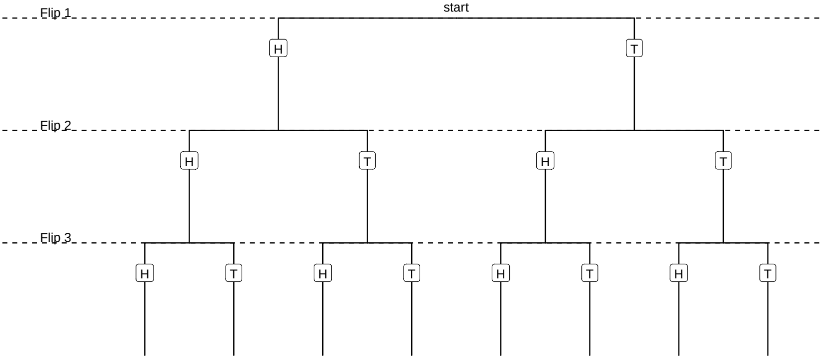
How did they calculate the expected winnings?

- The challenge was to enumerate all possible outcomes of the game and determine their frequency.
- A coin flip is a type of experiment. A series of coin flips form a multistage experiment.
  - Outcomes of small multistage experiments are represented visually with a tree diagram, which we demonstrate here.
- Pascal devised an algorithm for handling a large number of independent coin flips using his eponymous triangle.
  - Fermat figured out why the algorithm worked.
  - The results were published posthumously in Treatise on Arithmetical Triangle (1665).
  - In this lecture, we use tree diagrams because the number of coin flips is small enough to completely enumerate all outcomes on one slide.

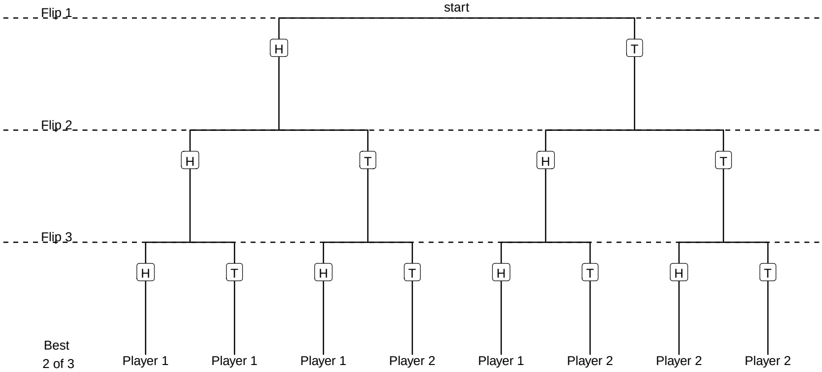
Step 1: Enumerate all possible outcomes



Player 1 expected winnings = ?

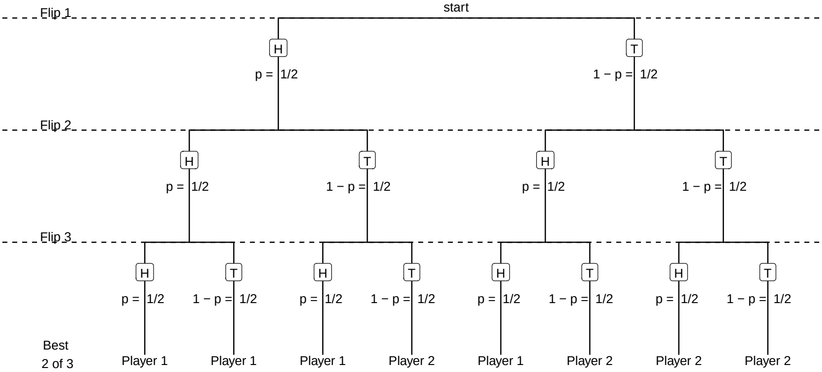
Player 2 expected winnings = ?

Step 1: Enumerate all possible outcomes



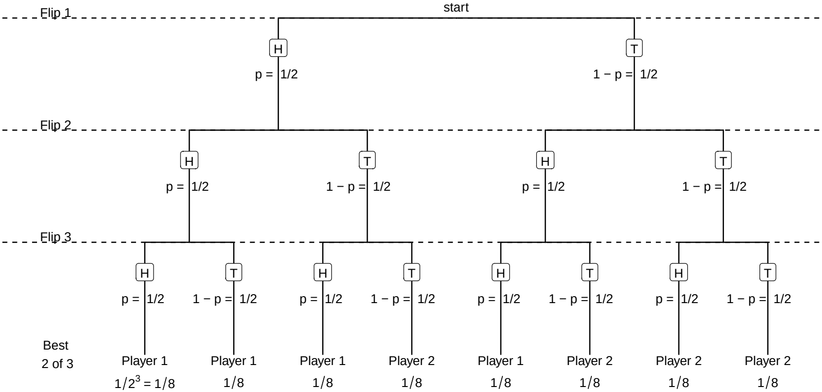
Player 1 expected winnings = ?
Player 2 expected winnings = ?

Step 2: Label the probability of outcomes by stage



Player 1 expected winnings = ?
Player 2 expected winnings = ?

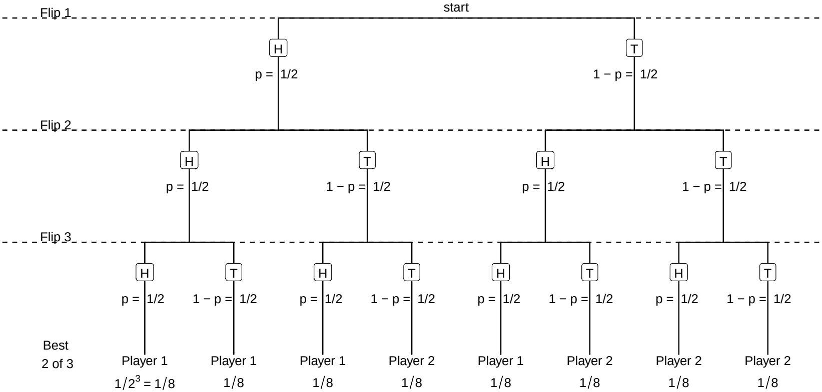
Step 3: Multiply vertical probabilities



Player 1 expected winnings = ?

Player 2 expected winnings = ?

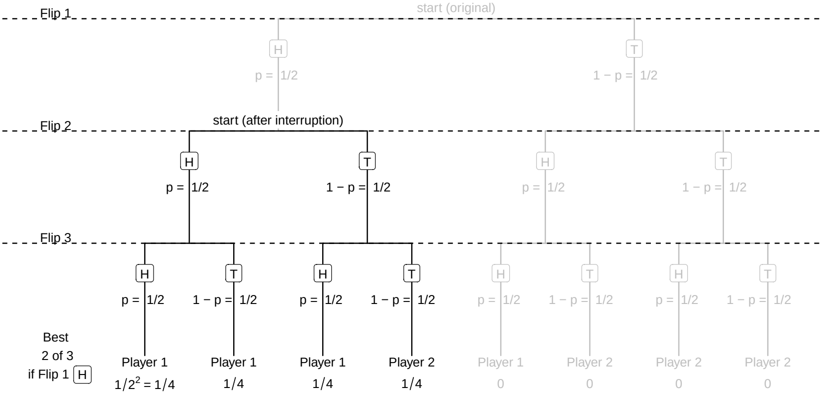
Step 4: Add probability-weighted outcomes



Player 1 expected winnings = 1/8 x \$100 + 1/8 x \$100 + 1/8 x \$100 + 1/8 x \$100 = \$50

Player 2 expected winnings = 1/8 x \$100 + 1/8 x \$100 + 1/8 x \$100 + 1/8 x \$100 = \$50

Solution for best 2 of 3 if Player 1 ahead by 1



Player 1 expected winnings = 1/4 x \$100 + 1/4 x \$100 + 1/4 x \$100 = \$75

Player 2 expected winnings = 1/4 x \$100 = \$25

Pascal's table for best 3 of 5 if Player 1 ahead by 1 (Tannery 1894)

the given points, and leaves on the planes segments in which given angles may be inscribed;"[1] and this one: "Any three circles, any three points, and any three lines being given, to find a circle which touches the circles and the points and leaves on the lines an arc in which a given angle may be inscribed."

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ular cases when it is shorter than the general method), a method which is good only in isolated cases and not good for others.

I am sure that I can make it understood, but it requires a few words from me and a little patience from you.

2. This is the method of procedure when there are *two* players: If two players, playing in several throws, find themselves in such a state that the first lacks *two* points and the second *three* of gaining the stake, you say it is necessary to see in how many points the game will be absolutely decided.

It is convenient to suppose that this will be in *four* points, from which you conclude that it is necessary to see how many ways the four points may be distributed between the two players and to see how many combinations there are to make the first win and how many to make the second win, and to divide the stake according to that proportion. I could scarcely understand this reasoning if I had not known it myself before; but you also have written it in your discussion. Then to see how many ways four points may be distributed between two players, it is necessary to imagine that they play with dice with two faces (since there are but two players), as heads and tails, and that they throw four of these dice (because they play in four throws). Now it is necessary to see how many ways these dice may fall. That is easy to calculate. There can be *sixteen*, which is the second power of *four;* that is to say, the square. Now imagine that one of the faces is marked *a*, favorable to the first player. And suppose the other is marked *b*, favorable to the second. Then these four dice can fall according to one of these sixteen arrangements:

a	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*
a	*a*	*a*	*a*	*b*	*b*	*b*	*b*	*a*	*a*	*a*	*a*	*b*	*b*	*b*	*b*
a	*a*	*b*	*b*	*a*	*a*	*b*	*b*	*a*	*a*	*b*	*b*	*a*	*a*	*b*	*b*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*
1	1	1	1	1	1	1	2	1	1	1	2	1	2	2	2

and, because the first player lacks two points, all the arrangements that have two *a*'s make him win. There are therefore 11 of these for him. And because the second lacks three points, all the arrangements that have three *b*'s make him win. There are 5 of these. Therefore it is necessary that they divide the wager as 11 is to 5.

There is your method, when there are *two* players, whereupon you say that if there are more players, it will not be difficult to make the division by this method.

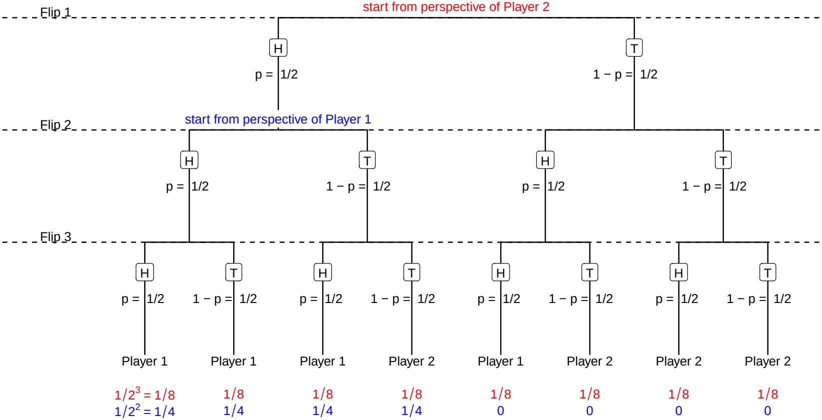
In what sense is the expected value fair?

- A player's winnings are representative, proportional to:
  - the frequency the player would win the game if played a large number of times, or equivalently
  - the probability the player would win the game if one of these playings were randomly chosen.

- Conversely, identical players winning at equal rates (or two groups of players winning similar amounts) suggest the game is fair.
  - Traditionally, equal division called "unbiased". Unequal, "biased".

- The 1970 Supreme Court Case Griggs v. Duke Power Co. established this definition of fair as a basis for challenging employment selection procedures under Title VII of the 1964 Civil Rights Act (equal employment opportunity).
  - They ruled a fair algorithm should hire qualified White and Black applicants at equal rates.
  - Bias may not be intended (disparate treatment/inequity). Neutral policies can adversely affect one group (disparate impact/inequality).

But one's expectation often a matter of perspective!



Player 2 expects 1/2 of the pot
Player 1 expects 3/4 of the pot

Expectation (unbiasedness) just one notion of "fair"

- Calling an algorithm unbiased draws an analogy between a group of observations (e.g. people) and hypothetical replications of a game.
- When the outcome is binary (e.g. win or lose), the results are often communicated in percentages. For example,
  - Suppose Player 1 and Player 2 are identical except that for every 100 Player 1s that play the game, 75 win.
  - Since Player 1 and Player 2 are identical, Player 1 is only expected to win 50 percent of hypothetical replications of the game.
  - But Player 1 won 75 percent, suggesting game biased in favor of Player 1. The bias is a 75 - 50 = 25 percent advantage.
- But groups of people are not like repeated games. An unbiased algorithm does not consider:
  - whether one player values or needs the win more than another.
  - whether the outcome of one game is close to the outcome expected if the game were repeated.
  - whether the individuals within each group should be compared in the first place. e.g. Are Player 1 and Player 2 really identical?

References

1. Devlin, Keith. The unfinished game: Pascal, Fermat, and the seventeenth-century letter that made the world modern. Basic Books, 2010.

2. Meier, Paul, Jerome Sacks, and Sandy L. Zabell. "What happened in Hazelwood: Statistics, employment discrimination, and the 80% rule." American Bar Foundation Research Journal 9.1 (1984): 139-186.

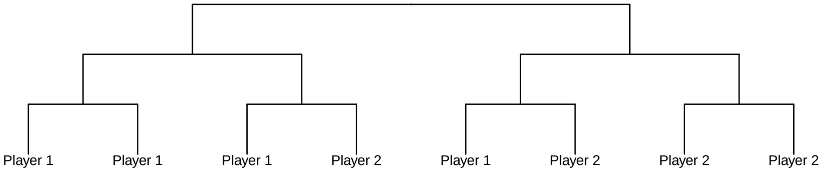
3. Oeuvres de Fermat, t. II, Correspondence, éd. P. Tannery et C. Henry, Paris, Gauthier-Villars, 1894.

4. Smith, David Eugene. A source book in mathematics. Courier Corporation, 2012.

Appendix: Newick Representation of Decision Tree

```r
# Tree coded using Newick format
## parens. denote grouping of terminal nodes
## c.f. https://en.wikipedia.org/wiki/Newick_format

tree_text <- "((('Player 1', 'Player 1'),
                ('Player 1', 'Player 2')),
              (('Player 1', 'Player 2'),
                ('Player 2', 'Player 2')));"
tree_data <- read.newick(text = tree_text)
ggtree(tree_data) + layout_dendrogram() +
  geom_tiplab(hjust = .5, vjust = 1, parse = TRUE)
```



Appendix: Graph a lightly annotated tree

```
(decision_tree_unlabeled <-
    ggtree(tree_data) +
    layout_dendrogram() +
    geom_vline(xintercept = -(1:3), linetype = 2) +
    annotate("text", x = -(1:3)-.05, y = 0,
             label = paste("Flip", 3:1)) +
    annotate("text", x = -3.1, y = 4.5, label = "start") +
    geom_label2(aes(branch,
                    label = c(rep(c("H","T"), 4), NA,
                              "H", "H", "T", "T", "H", "T")),
             vjust = -1) +
    annotate("text", x=0, y=0, label = "Best\n 2 of 3") +
    geom_tiplab(hjust = .5, vjust = 1, parse = TRUE))
```

Appendix: Graph an annotated tree

```
(decision_tree_labeled <-
    decision_tree_unlabeled +
    geom_text(aes(branch),
      label = c(rep(c("p =  1/2", "1 - p =  1/2"),4), NA,
                rep("p =  1/2 ",2), rep("1 - p =  1/2",2),
                "p =  1/2 ", "1 - p =  1/2")) +
    annotate("text", x = .25, y = 2:8, label = "1/8",
             parse = TRUE) +
    annotate("text", x=.25, y=1, label = "1/2^3~'='~1/8",
             parse = TRUE) +
    annotate("text", x = 1, y = 4.5,
             label = "Player 1 wins with frequency =
             1/8 + 1/8 + 1/8 + 1/8 = 1/2
           Player 2 wins with frequency =
           1/8 + 1/8 + 1/8 + 1/8 = 1/2") +
    annotate("text", x = 1.5, y = 4.5,
             label = "Player 1 expected winnings =
             1/2 of pot"))
```